A. What you thought 25%.
B. Better than exp. 25%.
C. Worse than you exp. 48%.

CSE 150A-250A AI: Probabilistic Models

D. Test2 Scores are out? 2%.

Lecture 17

Fall 2025

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Slides adapted from previous versions of the course (Prof. Lawrence, Prof. Alvarado, Prof Berg-Kirkpatrick)

Agenda

Review

Exploration vs Exploitation

Reinforcement Learning

- Stochastic approximation theory
- Temporal difference prediction

A. Milestone 1 / Milestone 2 or spoke TA

B. You haven't.

Review

Policy Improvement

- **Greedy policy:**

$$\pi'(s) \quad = \quad \underset{a}{\text{argmax}}\; Q^{\pi}(s, a)$$

- **Theorem:**
  The greedy policy $\pi'(s) = \arg\max_a Q^{\pi}(s, a)$ improves everywhere on the policy π from which it was derived:

$$V^{\pi'}(s) \; \geq V^{\pi}(s) \quad \text{for all states } s \in \mathcal{S}$$

How to compute π^*?

1. Choose an initial policy $\pi : \mathcal{S} \to \mathcal{A}$.
2. Repeat until convergence:

   Compute the action value function $Q^{\pi}(s, a)$.
   Compute the greedy policy $\pi'(s) = \text{argmax}_a Q^{\pi}(s, a)$.
   Replace π by π'.

$$\pi_0 \xrightarrow{\text{evaluate}} \begin{array}{c} V^{\pi_0}(s) \\ Q^{\pi_0}(s,a) \end{array} \xrightarrow{\text{improve}} \pi_1 \xrightarrow{\text{evaluate}} \begin{array}{c} V^{\pi_1}(s) \\ Q^{\pi_1}(s,a) \end{array} \xrightarrow{\text{improve}} \cdots$$

Value iteration

- Idea in a nutshell

  Replace the equality sign in the Bellman optimality equation by an assignment operation:

$$V^*(s) = \max_a \left[R(s) + \gamma \sum_{s'} P(s'|s,a)\, V^*(s') \right] \quad \boxed{\text{BOE}}$$

$$V_{\text{new}}(s) \longleftarrow \max_a \left[R(s) + \gamma \sum_{s'} P(s'|s,a)\, V_{\text{old}}(s') \right] \quad \boxed{\text{algorithm}}$$

Algorithm for value iteration

1. **Initialize:** $V_0(s) = 0$ for all $s \in \mathcal{S}$.

2. **Iterate until convergence:**

$$V_{k+1}(s) \;=\; \max_a \left[R(s) \;+\; \gamma \sum_{s'} P(s'|s,a)\, V_k(s') \right] \text{ for all } s \in \mathcal{S}.$$

3. **Solve for optimal policy**:

$$\begin{aligned} Q_k(s,a) &= R(s) + \gamma \sum_{s'} P(s'|s,a)\, V_k(s'), \\ \pi^*(s) &= \lim_{k \to \infty} \operatorname{argmax}_a Q_k(s,a). \end{aligned}$$

Value iteration (VI) versus policy iteration (PI)

- **Compare and contrast:**

  PI searches through the combinatorial space of policies.
  VI searches through the continuous space of value functions.

- **Convergence:**

  PI converges in a finite number of steps.
  VI converges asymptotically (in the limit).

Exploration vs Exploitation


GOLD CHAMPION
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Slot Machine!
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Multi-Armed Bandit

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Multi-Armed Bandit

- Stateless MDP: N one-armed bandits.
- Each bandit pays a random reward from an unknown probability distribution. Some bandits are more likely to get a winning payoff than others.
- **Goal**: Maximize the total rewards of a sequence of lever pulls.

Definition: A **multi-armed bandit** is defined by a set of random variables R_{at} where:

- $1 \leq a \leq N$, such that a is the arm of the bandit; and
- t the index of the play of arm a;

Multi-Armed Bandit

Definition: A **multi-armed bandit** is defined by a set of random variables R_{at} where:

- $1 \leq a \leq N$, such that a is the arm of the bandit; and
- t the index of the play of arm a;

Successive plays are assumed to be independently distributed, but we do not know the probability distributions of the random variables.

MAB - Action Value

Action value can be estimated:

$$Q(a) = \frac{1}{N(a)} \sum_{t=1}^{T} R_{at}$$

where t: number of rounds so far,

$N(a)$: number of times a was selected in previous rounds

R_{at} : reward obtained in the round t for playing arm a.

Exploitation vs Exploration dilemma

Goal: Maximize the reward

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- Ideally, keep playing the actions that have given us the **best** reward.

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- We want strategies that **exploit** what we think are the best actions so far, but still **explore** other actions.

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Goal: Maximize the reward

- Ideally, keep playing the actions that have given us the **best** reward.
- Initially, we do not have enough information to tell us what the best actions are.
- We want strategies that **exploit** what we think are the best actions so far, but still **explore** other actions.

But how much should we exploit and how much should we explore? This is known as the exploration vs. exploitation dilemma.

$\epsilon-$first strategy

Explore the options uniformly for some time, and then once we are confident we have enough samples (when the changes to the $Q(a)$ of start to stabilize), we **exploit** $argmax_a Q(a)$.

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Can we do better?

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Can we do better? Time is wasted equally in all actions using the uniform distribution. Instead, we can focus on the most promising actions given the rewards we have received so far.

$\epsilon-$greedy strategy

With some probability, $\epsilon \in [0, 1]$

- Choose a random arm with uniform probability. Update $Q(a)$.

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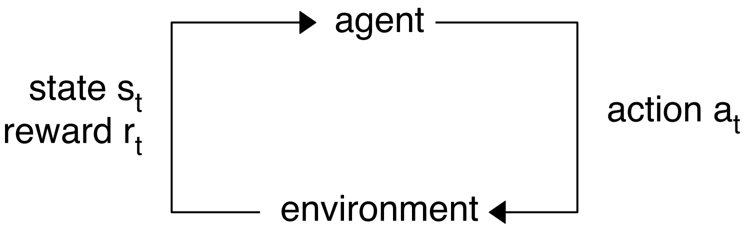
With some probability, $\epsilon \in [0, 1]$

- Choose a random arm with uniform probability. Update $Q(a)$.

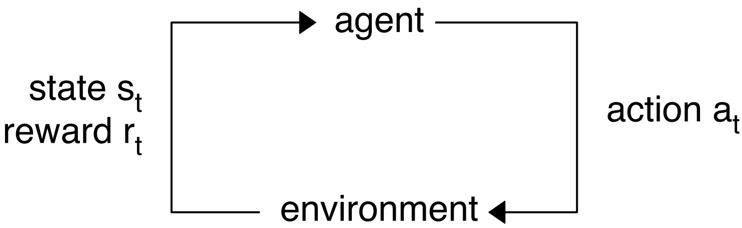
With probability, 1 - ϵ

- Choose arm with maximum action value: $argmax_a Q(a)$

Reinforcement Learning


agent
state s_t
reward r_t
action a_t
environment

Reinforcement learning



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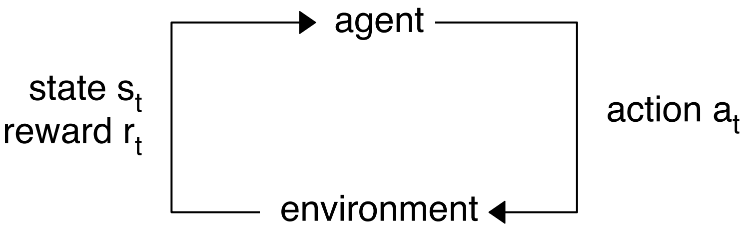
Reinforcement learning



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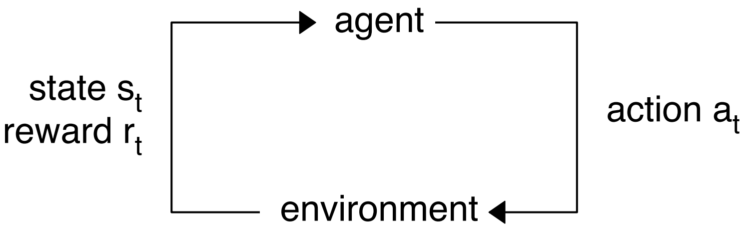

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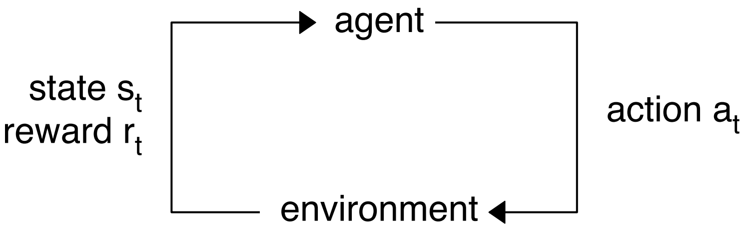

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But what if we don't know $P(s'|s,a)$ and $R(s)$?

Can we learn an optimal policy *directly from experience*?

Model-based approach

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- Estimate model from experience

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  Explore world and estimate $\hat{P}(s'|s,a) \approx P(s'|s,a)$ from samples.

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Is it really necessary to estimate a model?

Model-free approach

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- **But for this we need new tools:**

  Stochastic approximation theory
  Temporal difference (TD) learning

Taking Averages Sample by Sample

Let's say I'm playing a game where I can score between 1 and 10 points. What would you predict my score would be the next time I play it? What if you knew that in the past, I have scored these scores (not necessarily in this order)

8, 8, 2, 5, 7, 2, 5, 7, 1, 3

What score would you predict I will get?

A. 3

B. 5

C. 7

D. 9

E. 10

Stochastic approximation theory

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How to estimate the mean of a random variable X from IID samples?

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1. Sample average

$$\mu_T \ = \ \frac{1}{T}\big(x_1 + x_2 + x_3 + \cdots + x_T\big)$$

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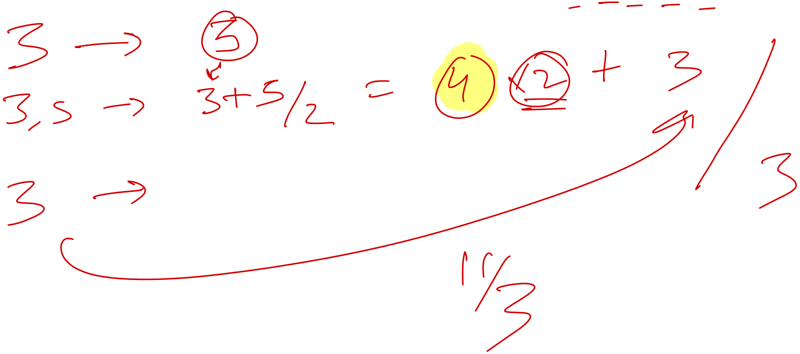
This is the most obvious estimate, but not the only one …

Let’s average these numbers, in 5 iterations: 3, 5, 3, 8, 10



Taking averages, sample by sample

Let's average these numbers, in 5 iterations: 3, 5, 3, 8, 10

1. Score 3, Avg: 3

Taking averages, sample by sample

Let's average these numbers, in 5 iterations: 3, 5, 3, 8, 10

1. Score 3, Avg: 3
2. Score 5, Avg: $(1/2)3 + (1/2)5 = 4$

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Let's average these numbers, in 5 iterations: 3, 5, 3, 8, 10

1. Score 3, Avg: 3
2. Score 5, Avg: (1/2)3 + (1/2)5 = 4
3. Score 3, Avg: (2/3)4 + (1/3)3 = (1 - 1/3)4 + (1/3)3 = 11/3

$1 - 1/3$

$= 2/3$

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$$\mu_t = \mu_{t-1} + \alpha_t(x_t - \mu_{t-1})$$

Let's say I'm playing a game where I can score between 1 and 10 points. What would you predict my score would be the next time I play it? What if you knew that in the past, I have scored these scores in **this order**:

1, 2, 3, 2, 5, 7, 5, 7, 8, 8

Is your guess about my next score higher, lower or the same as last time (5)?

A. Higher

B. Lower

C. The same

Stochastic approximation theory (con't)

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    Initialize: $\mu_0 \quad = \quad 0$

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$$\mu_t = \mu_{t-1} + \alpha_t(x_t - \mu_{t-1})$$

The corrective term $\boxed{x_t - \mu_{t-1}}$ is known as a **temporal difference**.

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2. **Incremental update**

$$\begin{aligned} \textbf{Initialize:} \quad \mu_0 &= 0 \\ \textbf{Update:} \quad \mu_t &= (1-\alpha_t)\mu_{t-1} + \alpha_t x_t \qquad \text{for} \quad \alpha_t \in (0,1) \end{aligned}$$

   The update is a convex sum of the old estimate and latest sample.
   It can also be written as:

$$\mu_t = \mu_{t-1} + \alpha_t(x_t - \mu_{t-1})$$

   The corrective term $\boxed{x_t - \mu_{t-1}}$ is known as a **temporal difference**.
   This is the simplest example of a temporal difference (TD) update.

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(i) α_t decays sufficiently slowly to incorporate many examples
(ii) α_t decays sufficiently fast to converge in the limit

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That's all folks!